Adaptive Extremum Control Using Approximate Process Models

The proposed on-line adaptive optimization technique incorporates *a priori* knowledge in the form of approximate steady-state models. The steady-state geometric characteristics of the model are periodically recalculated using a Hammerstein system and recursive least squares. The algorithm is self-tuning in the sense that it converges to the optimal performance provided that a matching condition is satisfied and that the data are persistently exited. Simulation and experimental studies performed on a continuous fermentation system have been conducted to illustrate the performance of the optimization algorithm and demonstrate the viability of adaptive extremum control.

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Introduction

The objective of an adaptive extremum controller is to locate the steady-state optimum of a process and then continuously keep the process operating at its optimum despite inaccuracies in the model, drifts due to unmeasured process disturbances, and slow dynamical changes that result from changing process parameters. A number of different techniques based on gradient estimation for solving this problem are reviewed by Arkun and Stephanopoulos (1980), Sternby (1980), and Garcia and Morari (1981). Model-oriented optimization schemes, which use adaptive estimation, solve the problem of model inaccuracies and uncertainties by identifying the parameters of a discrete dynamic Hammerstein model of the process. The Hammerstein representation is linear in terms of the unknown parameters of the system and on-line identification techniques such as recursive least squares, approximate maximum likelihood, and instrumental variables apply. The steady-state, input-output model of the process, which is obtained from the estimated dynamic model, is then used to find the optimal input sequence.

A few applications of this type of optimization procedure have been described in the literature. For example, Bamberger and Isermann (1978) describe an on-line Newton's method for the optimization of multiinput, multioutput systems, which requires the identification of a generalized second-order Hammerstein model. This algorithm was used to maximize the power produced from a steam turbine. Lee and Lee (1985) used the approach of Bamberger and Isermann to develop a cascade structure for the simultaneous optimization and control of a fixed bed reactor. The multivariable optimizing controller consisted of two loops:

1) An outer loop which identifies a multiinput, single-output process and performs a constrained optimization using a random search technique

2) An inner loop which adaptively controls the bed temperature based on a setpoint supplied by the outer loop

An approach based on linear models and a steepest descent search was used by Garcia and Morari (1981) for the maximization of the product of a reversible reaction in a series reactor system. They later interfaced this optimization technique with a multivariable regulator to ensure robustness and smooth transfer between operating conditions (Garcia and Morari, 1985). A similar technique was also used by Rolf and Lim (1984, 1985) for the maximization of the production of Bakers' Yeast in a continuous bioreactor. In general, steepest descent algorithms converge slowly for most problems. Harmon et al. (1987) proposed a "cautious" adaptive optimizer. Their technique requires the identification of a linear model, but does not rely on a gradient technique for computing the optimal input. Rather, the steady-state model obtained is substituted directly into the performance index which is then solved for the optimal input. This input is then "cautiously" implemented using a variable step size. All of the algorithms mentioned above show good asymptotic properties, and simulation and experimental results indicate that the optimum is reached in a reasonable amount of time. In fact, Svoronos and Lyberatos (1987) show that all the methods discussed above have good stability and convergence properties, provided that the model structure is rich enough to represent the measured process data exactly.

In this paper we extend the method due to Bamberger and Isermann to include a priori knowledge of the static nonlinear

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behavior of the process. In this way we are able to incorporate a realistic model structure which in a qualitative manner models the process data over a wider range of operating conditions than is achievable with a linear or quadratic Hammerstein model. The optimizer uses a set of modified equality constraints constructed so that they have the same geometric characteristics at the current operating point as the true steady-state behavior of the process. However, they are based structurally upon simple relationships of the static behavior of the process (i.e., a design equation). Such relationships exist for the majority of chemical processes and they usually give a good approximation of the structural properties of the process. Subject to persistent excitation and a matching condition we show that the algorithm converges in such a way that the modified equality constraints represent the local behavior of the process sufficiently accurately to ensure convergence to the true steady-state optimum.

The paper includes a discussion on the stability and convergence of the proposed algorithm and an application of the proposed optimization procedure to a continuous culture system for the purpose of maximizing biomass. This is the problem studied by Rolf and Lim (1984, 1985), and Harmon et al. (1987). The system is an ideal candidate for on-line model-oriented optimization for several reasons including:

1. Accurate kinetic models of microbial systems are difficult to obtain. Thus, most optimization schemes for these systems have been based on steady-state data obtained off-line. Obtaining such data can be a time-consuming procedure since the process settling times for fermentors can be 20 or more hours in some operating regions.

2. The optimal operating point depends strongly on the conditions of the feed, the culture, and the environment. Thus, it may change or drift according to changes or disturbances in these conditions. Such changes in the optimum could thereby go unnoticed, which means the system would be operating suboptimally, or if the change is noticed and/or premeditated, this would mean that steady-state data would have to be collected at the new operating conditions so that the process could be reoptimized. An on-line, model-based adaptive optimizer solves this problem by continually updating the process model.

We conclude the paper by discussing the influence of the different tuneable parameters on the performance of the algorithm.

Nonlinear Optimization Using Approximate Models

The steady-state optimization problem for a single-input, single-output process is given by,

$$\max_u J = L(y,u,\psi) \tag{1a}$$

subject to

$$p(y,u,\psi) = y - f(u,\psi) = 0 \tag{1b}$$

where

y = process output
u = manipulated process input
ψ = a vector of parameters and process variables
L = objective function
f = equality constraint

If the equality constraint given by Eq. 1b is known exactly, then the static optimization problem can be solved directly using Lagrange multipliers. However, for many processes it is difficult to find an exact analytic expression for the steady-state behavior of the process, and many process parameters and disturbance variables may not be known accurately. Thus, the optimization problem cannot be accurately represented and direct optimization leads to the wrong result. This problem is serious especially if the optimum operating conditions are close to a constraint as they are in many bioapplications. For such processes, it is advantageous to consider adaptive extremum control and improve the accuracy of the optimization by using successive approximation.

For the purpose of developing the theory, we assume that we have an approximation to the steady-state equation (Eq. 1b), which is given by,

$$\hat{p}(y,u,\hat{\psi}) = y - \hat{f}(u,\hat{\psi}) - \Delta = 0 \tag{2}$$

where

$\hat{f}$ = approximate equality constraint
$\hat{\psi}$ = subset of parameters and process variables
$\Delta = f - \hat{f}$; the model error

The nonlinear function $\hat{f}$ may, for example, be obtained from a steady-state design relationship. Although $\hat{f}$ only approximates f in the sense that the error, Δ, may be quite large in some operating regions, it is important to recognize that the steady-state design equation for the process gives some structural information about the static nonlinear nature of the process. It is this property we exploit in this paper. Note that if $\hat{f} = 0$, then no prior knowledge is assumed about the system and the modeling error then equals the equality constraint used in Eq. 1b. The algorithm proposed in this paper reduces to the Newton's method technique developed by Bamberger and Isermann (1978) in this particular case.

The aim of the analysis that follows is to use a simple idea from differential geometry and match parameterized input-output data to the approximate representation given by Eq. 2. The matching process proceeds in a manner that constructs a well-behaved, "modified" equality constraint which accurately represents the nonlinear nature of the process locally about the current operating point. To do this, we define a new constraint

$$y - F(u,u^*,\hat{\psi},\psi^*) = 0 \tag{3}$$

where

u^* = current process input
ψ^* = current process operating conditions

We now require that $F(.)$ have the same local geometric characteristics at the current operating point as the unknown function (Eq. 1b). The relationships between the true, approximate, and modified equality constraints are illustrated in Figure 1.

The first geometric criterion for the construction is the equality of the curvatures at the current operating point. Since we

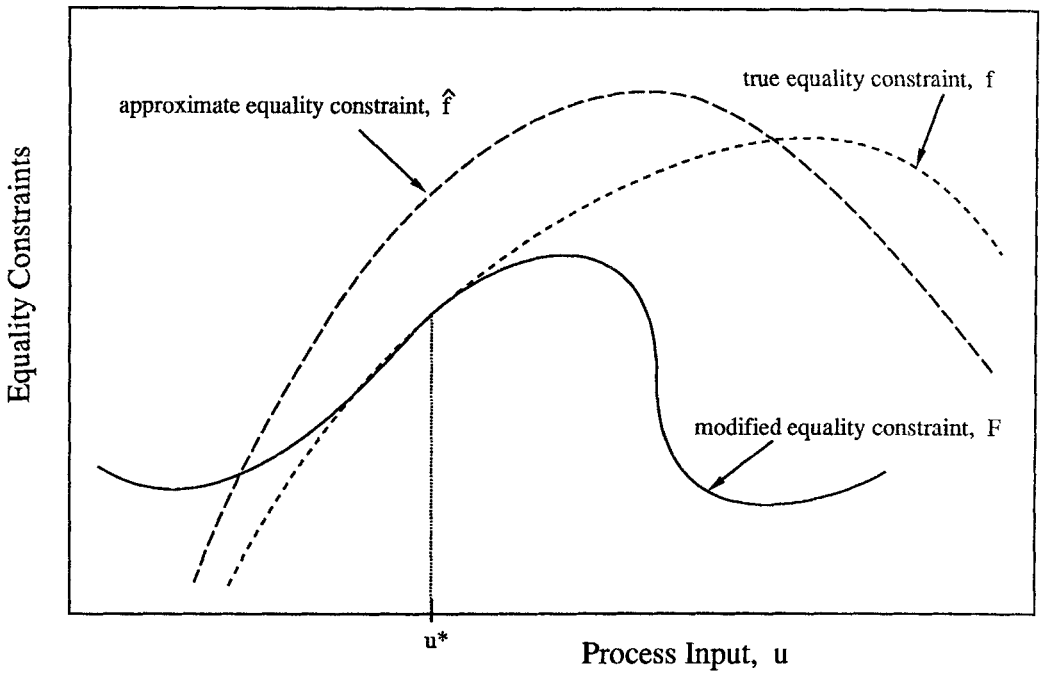

Figure 1. Relationships among true, approximate and modified equality constraints.

also require equality of the tangents, this criterion reduces to

$$\left.\frac{d^2F}{du^2}\right|_{u^*,\psi^*} = \left.\frac{d^2f}{du^2}\right|_{u^*,\psi^*} \tag{4}$$

The equality given by Eq. 4 is met by,

$$\frac{d^2F}{du^2} = \frac{d^2\hat{f}}{du^2} + \left.\frac{d^2f}{du^2}\right|_{u^*,\psi^*} - \left.\frac{d^2\hat{f}}{du^2}\right|_{u^*,\psi^*} \tag{5}$$

at the point, $u = u^*$. Integrating Eq. 5 twice, we obtain the final expression for F

$$F(u,u^*,\psi,\psi^*) = \hat{f}(u,\hat{\psi}) - \hat{f}(u^*,\hat{\psi}) + \frac{k_1}{2}(u^2 - u^{*2}) + k_2(u - u^*) + f(u^*,\psi^*) \tag{6a}$$

where

$$k_1 = \left.\frac{d^2f}{du^2}\right|_{u^*,\psi^*} - \left.\frac{d^2\hat{f}}{du^2}\right|_{u^*,\psi^*} \tag{6b}$$

and k_2 is the constant of integration. To evaluate k_2, we use the criterion of equality of the tangents at the current operating point,

$$\left.\frac{dF}{du}\right|_{u^*,\psi^*} = \left.\frac{df}{du}\right|_{u^*,\psi^*}. \tag{7}$$

We can see from Eq. 6a that the final geometric criterion, equality of the function values at the current operating point, is satisfied when $u = u^*$. The quantities $d^if/du^i|_{u^*,\psi^*}$ $i = 0,1,2$ are not known accurately. However, they can be estimated by using the following approach.

First, we use the fact that in a small neighborhood about the operating point, the steady-state, input-output response can be represented as a second order Taylor series

$$f(u,\psi) = f(u^*,\psi^*) + \left.\frac{df}{du}\right|_{u^*,\psi^*}(u - u^*) + \left.\frac{d^2f}{du^2}\right|_{u^*,\psi^*}\frac{(u - u^*)^2}{2!} + \Omega \tag{8}$$

where Ω is the remainder. Then we represent the local dynamics of the process by a discrete time Hammerstein model,

$$A(q^{-1})y(t) = B(q^{-1})u(t-1) + C(q^{-1})u^2(t-1) + v + R(t) \tag{9}$$

where v is the process bias and A, B, and C are polynomials of orders n and m, respectively, in the backward shift operator, q^{-1}, such that

$$A(q^{-1}) = 1 + a_1q^{-1} + a_2q^{-2} + \cdots + a_nq^{-n}$$
$$B(q^{-1}) = b_0 + b_1q^{-1} + b_2q^{-2} + \cdots + b_mq^{-m}$$
$$C(q^{-1}) = c_0 + c_1q^{-1} + c_2q^{-2} + \cdots + c_mq^{-m}$$

and $R(t)$ denotes unrepresented dynamics and nonlinearities. It is necessary that A and C are stable or stabilized using feedback control. For a given operating point, A, B, and C are constant so the static behavior of the discrete process, can be obtained by invoking the final value theorem for z-transforms, i.e.

$$A(1)y(t) = B(1)u(t) + C(1)u^2(t) + v + R(1) \tag{10}$$

Comparing the two steady-state process models given by Eqs. 8

and 10, we obtain that

$$\left.\frac{d^2f}{du^2}\right|_{u^*,\psi^*} = 2\frac{C(1)}{A(1)}, \tag{11a}$$

$$\left.\frac{df}{du}\right|_{u^*,\psi^*} = 2\frac{C(1)}{A(1)}u^* + \frac{B(1)}{A(1)}, \tag{11b}$$

and

$$f(u^*,\psi^*) = \frac{C(1)}{A(1)}u^{*2} + \frac{B(1)}{A(1)}u^* + \frac{v}{A(1)}. \tag{11c}$$

By noting that the parameters in the Hammerstein model (Eq. 9) enter linearly in terms of the observable quantities u,u^2, and y, we can use standard identification techniques to obtain the estimates needed in expressions 11 a–c. This enables us to solve the "modified" static optimization problem on-line at the current operating point

$$\max_u \hat{J} = L(u,F(u,u^*,\hat{\psi},\psi^*)) \tag{12}$$

The computed input value may not be the true optimum of the system, so this procedure is repeated at each operating point. As with most optimization methods based on search techniques, we can only guarantee that we will converge to the global optimal solution if we begin operating in the convex region about the optimum.

Parameter Estimation and Convergence

To estimate the unknown parameters of the local process model, we rewrite Eq. 9 as

$$y(t) = \theta^T\varphi(t-1) + R(t) \tag{13}$$

where

$$\varphi^T(t-1) = \{y(t-1),\ldots,y(t-n),u(t-1),\ldots,u(t-m-1),$$
$$u^2(t-1),\ldots,u^2(t-m-1),1\}$$
$$\theta^T = \{-a_1,-a_2,\ldots,-a_n,b_0,b_1,\ldots,b_m,c_0,c_1,\ldots,$$
$$c_m,v\}.$$

The parameters, $\{a_i,b_i,c_i\}$, are unknown and generally time-varying since the process model is only valid locally about the current operating point. We use a weighted recursive least-squares estimator with variable forgetting to obtain time-varying estimates $\hat{\theta}(t)$ of θ that minimize the residuals $R(t)$, the unmodelled term, under the l_2 norm recursively. In fact, recursive least squares solves the problem

$$N(t) = \min_\theta \sum_{i=t-N}^{t} w(i)(y(i) - \varphi(i-1)^T\theta)^2$$
$$\text{(data from segment } [t-N,t])$$
$$+ w(t-N-1)[\hat{\theta}(t-N) - \theta]^T$$
$$\cdot P(t-N)^{-1}[\hat{\theta}(t-N) - \theta]$$
$$\text{(initial conditions)} \tag{14}$$

where $\{w(i),i = t-N-1,\ldots,t\}$ is a sequence of positive weights and $P(t-N)$ is a matrix which will be defined later. One way of interpreting the estimation problem in light of expression 14 is that the estimate $\hat{\theta}(t)$ minimizes the errors over a finite segment $[t,t-N]$, while including a finite weight on the initial estimate provided through $\hat{\theta}(t-N)$. Thus, the effect of initial conditions can be made small by choosing $|P(t-N)^{-1}|$ small. If the "true" parameters $\theta(t)^o = \theta^o$ are constant over the same segment of data, then clearly

$$N(t) \le \sum_{i=t-N}^{t} w(i)R(i)^2$$
$$+ w(t-N-1)[\hat{\theta}(t-N) - \theta^o]^TP(t-N)^{-1}[\hat{\theta}(t-N) - \theta^o]$$

We now make the following assumption:

Assumption A1. For a given data segment $\{y(i),\varphi(i-1),$ $i = t-N,t-N+1,\ldots,t\}$ there exists a constant vector θ^o so that for each $i(y(i) - \varphi(i-1)^T\theta^o)^2 \le \delta$, where δ is a positive number. If θ^o exists that sets $\delta \equiv 0$ over the entire interval then we get *exact matching.* This property is never met in practice. However, it is not unreasonable to assume that $|R(t)|$ is small if the process is operated close to steady state.

The weights $w(i)$ can be chosen in any number of different ways. One possibility consists of updating the weights so that

$$w(t) = r(t),\ w(t-i) = \prod_{j=0}^{i-1}\lambda(t-j)r(t-i) \quad 1 < i < N+1$$

where $r(t-i)$ is an estimate of the measurement variance associated with datapoint i and $0 < \lambda(i) < 1$ are the variable forgetting factors. This approach works well when the data are persistently excited. We then get the following algorithm.

Algorithm 1: Recursive Least Squares with Forgetting Factor

Given:

$$\{n,m,P(0) > 0,\hat{\theta}(0),\varphi(0),r(t)$$
$$\dim(\theta,\varphi) = (n+2m+3),$$
$$\dim(P) = (n+2m+3,n+2m+3)\}$$

Set:

$$t = 1$$

Compute:

1. $e(t) = y(t) - \hat{\theta}(t-1)^T\varphi(t-1)$
2. $P(t) = \lambda(t)^{-1}[P(t-1) - P(t-1)\varphi(t-1)\varphi(t-1)^TP(t-1)$ $/(r(t)\lambda(t) + \varphi(t-1)^TP(t-1)\varphi(t-1))]$
3. $\hat{\theta}(t) = \hat{\theta}(t-1) + P(t)\varphi(t-1)e(t)/r(t)$
4. Set $t = t+1$ and go to step 1.

Figure 2 gives a general block diagram of the on-line optimization procedure, showing the interaction of the estimation and optimization algorithms. In the algorithm, $P(t)$ is a symmetric positive definite matrix called the covariance matrix. It can, under certain conditions, be interpreted as a measure of the uncertainty of the parameters in which case it is defined as

$$P(t) = E[\tilde{\theta}(t)\tilde{\theta}(t)^T]$$

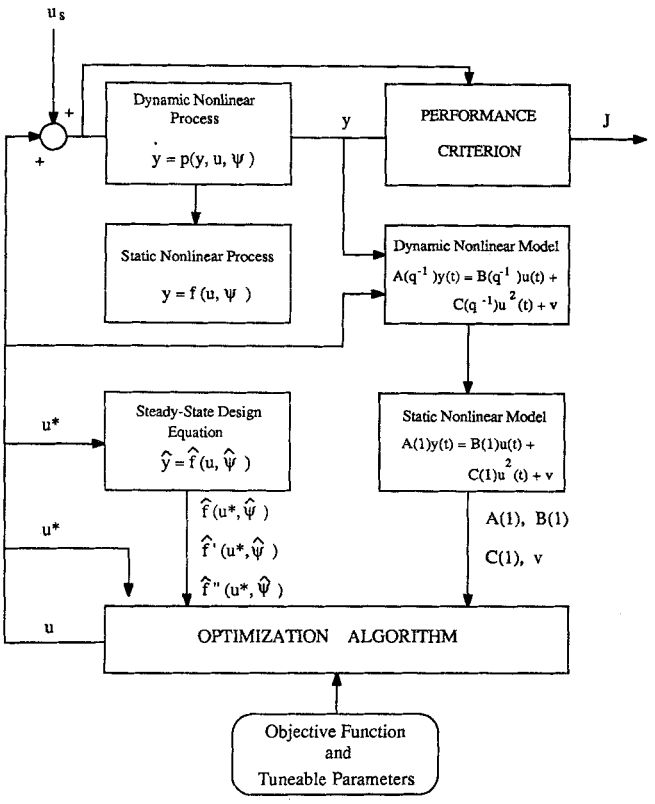

Figure 2. On-line optimization procedure.

where $\tilde{\theta}(t) = \theta - \hat{\theta}(t)$. This relation is exact in the Kalman filter. In the case of parameter estimation, it is true only if the data are persistently excited and the disturbance sequence is delta correlated. The inverse of the covariance matrix can be thought of as an information matrix. This can be seen from the update of $P(t)^{-1}$ which is given by

$$P(t)^{-1} = \lambda(t)P(t-1)^{-1} + \varphi(t-1)\varphi(t-1)^T/r(t). \quad (15)$$

Thus, each new piece of information which enters the regression vector, $\varphi(t-1)$, also affects the matrix $P^{-1}(t)$. Using the matrix inversion lemma, Eq. 15 is transformed to the update of the covariance matrix, $P(t)$, used in the estimation algorithm.

The inclusion of the forgetting factor enables the algorithm to adjust the parameter estimates of a time-varying process by limiting the size of the information matrix, $P(t)^{-1}$, given by Eq. 15. The forgetting factor may vary with time provided that it is limited so that $0 < \lambda_{\min} \leq \lambda(t) \leq 1$. The speed of adaptation is determined by the asymptotic memory length (Clarke and Gawthrop, 1975)

$$\Sigma(t) = 1/(1 - \lambda(t)) \quad (16)$$

which implies that the value of information decays with a time constant of $\Sigma(t)$ sample intervals. If $\lambda(t)$ equals one, then the estimator is not discounting past information and the asymptotic memory length is infinite. However, if $\lambda(t)$ is less than one the estimator is weighting past data exponentially with a finite asymptotic memory length.

We now introduce the notion of persistent excitation.

Definition D1. Suppose there exist finite positive numbers N, ϵ_1, ϵ_2 so that

$$\epsilon_1 I \leq \sum_{i=t-N}^{t} \varphi(i)\varphi(i)^T \leq \epsilon_2 I \quad \text{for all } t \geq N$$

then the vector sequence $\{\varphi(i), i = 0,1, \ldots t\}$ is said to be *persistently exciting of degree* $\{N,\epsilon_1,\epsilon_2\}$.

Theorem 1. Suppose that $\{\varphi(i), i = 0,1, \ldots t\}$ is persistently exciting of degree $\{N,\epsilon_1,\epsilon_2\}$, then the parameter error, $\tilde{\theta}(t) = \theta^o - \hat{\theta}(t)$, satisfies $\|\tilde{\theta}(t)\|^2 \leq c_1 R + c_2\|\tilde{\theta}(t-N)\|^2$ where $c_1 \to 0$ as $\epsilon_1 \to 0$ and $c_2 \to 0$ as $\epsilon_2 \to 0$ or $\gamma_{\min}(P(t-N)^{-1}) \to \infty$, where $\gamma_{\min}(.)$ denotes the minimum eigenvalue of a matrix. Moreover, if $\{\varphi(i), i = 0,1, \ldots t\}$ is persistently exciting of degree $\{N,\epsilon_1,\epsilon_2\}$, and the exact matching condition is satisfied then $\lim_{t,N\to\infty} \|\tilde{\theta}(t)\| = 0$.

Proof. See Appendix 1.

Theorem 1 states that if the residual $R(t)$ is bounded then the parameter error can be made small by increasing excitation and allowing the linear data segment to be sufficiently long. It is clear from this analysis that the optimization interval parameter N plays an important role in the optimization algorithm.

To reduce the number of tuneable parameters which must be chosen a priori, we have chosen to use the following update for $\lambda(t)$:

$$\lambda(t) = \max \left\{ \frac{\text{trace } P(t-1)}{\text{trace } P(t-1) + e(t)^2/(1 + \varphi(t-1)^T P(t-1)\varphi(t-1))}, \lambda_{\min} \right\} \quad (17)$$

where $0 \leq \lambda_{\min} \leq 1$ is a lower bound on the forgetting factor. If we compare this form of the forgetting factor update to that proposed by Fortescue et al. (1981), then we see that trace $P(t-1)$ is acting as a *time-varying* measure of the amount of information which is necessary to ensure good estimator sensitivity. This heuristic technique for updating the forgetting factor has proven fruitful in practical application (Kemna, 1987; Golden, 1988).

After the system has undergone an input change, $e(t)$ will be large since the previous parameter estimates give in general a poor local representation of the process at the new operating point. The value of trace $P(t-1)$ is small initially since it is based on the parameter uncertainty of the model at the previous operating point. Thus, the new forgetting factor will be small and past information is appropriately discarded. As information is discarded, trace $P(t-1)$ increases. Eventually, trace $P(t-1)$ becomes much greater than $e(t)^2/(1 + \varphi(t-1)^T P(t-1)\varphi(t-1))$. Now the forgetting factor will increase and information collected around the current operating point will no longer be discarded as rapidly. At some point, the forgetting factor becomes sufficiently large and trace $P(t-1)$ begins to decay to some small equilibrium value.

To get small parameter errors for the local second-order Hammerstein model, it is necessary to supply sufficient input excitation to the system in the form of a test signal superimposed on the input signal to the system. If the model were linear ($C(q^{-1}) = 0$), then a pseudorandom binary perturbation sequence would suffice. However, in the case of a quadratic model, a pseudorandom ternary perturbation sequence (PRTS) is needed to keep the covariance matrix nonsingular. This was

determined by Clarke and Godfrey (1966) and later confirmed by Bamberger and Isermann (1978). The PRTS sequence, $\{u_s\}$, is added directly to the input signal at each sampling period and will be randomly selected from the set $\{+u_{so}, 0, -u_{so}\}$.

The extremal control is computed periodically after a sufficiently long identification phase. In this way, the parameter estimates depend more on the current operating point than on past process operating conditions (Sternby, 1980) and we are assured good estimates of the steady-state information necessary for the optimization strategy. Theorem 1 gives some guidelines as to how long the optimization interval should be.

We now show that if we can achieve "perfect" estimation and the input sequence generated by the optimization algorithm converges, then the true optimum of the process is obtained.

Result 1. Suppose $\lim_{t\to\infty} |d^i(f - F)/du^i|_{u^*,\psi^*} = 0$ for $i = 0,1,2$, that $J(u)$ is convex and f and F are smooth to second order. If $\lim_{t\to\infty} \{u_i^o\} = u^o$ exists, then $\lim_{t\to\infty} \nabla_u J(u^o) = 0$.

Proof. See Appendix 1.

This leads directly to a *self-tuning* property of the adaptive optimizer.

Result 2. Suppose that the exact matching condition is satisfied for each $\{u^*(t), t > 0\}$ and that $\{\varphi(i), i = 0,1, \ldots t\}$ is persistently exciting of degree $\{N, \epsilon_1, \epsilon_2\}$. If $\lim_{t\to\infty} \{u_i^o\} = u^o$ exists, then u^o is the optimal steady state control.

Proof. Follows immediately from Theorem 1 and Result 1.

Application to a Continuous Bioreactor

The optimization procedure has been applied to the problem of maximizing the cellular productivity of a continuous culture system. The steady-state optimization problem is given as,

$$\max_D J = DX \tag{18a}$$

$$\text{s.t. } X - f(D,\psi) = 0 \tag{18b}$$

where the dilution rate, D, is the process input, the exit cell concentration, X, is the process output, and ψ represents the operating conditions of the culture. The criterion for a maximum is

$$\frac{dJ}{dD} = X + D\frac{dX}{dD} = 0 \tag{19a}$$

$$\frac{d^2J}{dD^2} = 2\frac{dX}{dD} + D\frac{d^2X}{dD^2} < 0. \tag{19b}$$

In practice the "true" steady-state operating curve, $X = f(D,\psi)$, is not known exactly due to the uncertain kinetics of the fermentation process. However, an approximate expression is given by the steady-state design equation for a continuous fermentor,

$$\hat{X} = \hat{f}(D,\hat{\psi}) \tag{20a}$$

Typically, the Monod equation or some other known microbial growth expression is used to define this equation which in the theory corresponds to the steady-state design equation represented by Eq. 2. By using the procedure described above we use on-line identification to obtain the modified equality constraint,

$$X - F(D,D^*,\hat{\psi},\psi^*) = 0. \tag{20b}$$

so that the static optimization problem becomes

$$\nabla_D \hat{J} = F(D,D^*,\hat{\psi},\psi^*) + D\frac{dF}{dD} = 0 \tag{21a}$$

and

$$\nabla_D^2 \hat{J} = 2\frac{dF}{dD} + D\frac{d^2F}{dD^2} < 0 \tag{21b}$$

where,

$$F(D,D^*,\hat{\psi},\psi^*) = \hat{f}(D,\hat{\psi}) - \hat{f}(D^*,\hat{\psi}) + \frac{k_1}{2}(D^2 - D^{*2}) + k_2(D - D^*) + f(D^*,\psi^*). \tag{21c}$$

We solve Eq. 21a for the optimum dilution rate and if it satisfies Eq. 21b, implement this as the new process input. This dilution rate may not be the optimum of the system, so this procedure is repeated at the next operating point.

Due to the physical constraint imposed by the phenomena of washout, the static functions, $\hat{f}$ and f, in general become singular at some value of the dilution rate. However, as long as $\hat{f}$ is smooth and nonsingular at least over the steady-state operating range of the fermentor, the optimization procedure is implementable.

Simulation Results

We will now present the simulation results of the nonlinear adaptive optimization which was carried out on the following dynamic fermentation process (Wang et al., 1979):

$$\dot{X} = (\mu - D)X \tag{22a}$$

$$\dot{S} = -\frac{\mu X}{Y} + D(S_o - S) - m_c X \tag{22b}$$

$$\mu = \frac{\mu_{max} S}{(K_s + S)} \tag{22c}$$

where

$\mu_{max} = 0.35\ h^{-1}$ (maximum specific growth rate)
$K_s = 0.09\ g/L$ (half-saturation constant)
$S_o = 5.0\ g/L$ (initial substrate concentration)
$Y = 0.5$ (yield coefficient)
$m_c = 0.025\ h^{-1}$ (maintenance coefficient)

The steady-state behavior of this process is given by

$$X = f(D,\psi) = \frac{YD}{m_c Y + D}\left(S_o - \frac{K_s D}{\mu_{max} - D}\right) \tag{23}$$

Notice that the cell maintenance term in the substrate balance makes the "observed" yield in Eq. 23 a function of the dilution rate at steady state. Also note that Eq. 23 becomes singular at $D = \mu_{max}$.

For the purpose of the optimization scheme, we suppose that

the steady-state design equation is given by

$$\hat{X} = \hat{f}(D,\hat{\psi}) = \hat{Y}\left(S_o - \frac{\hat{K}_s D}{\hat{\mu}_{max} - D}\right) \tag{24}$$

where

$\hat{Y} = 0.4$
$\hat{K}_s = 0.19\ g/L$
$\hat{\mu}_{max} = 0.42\ h^{-1}$
$S_o = 5.0\ g/L$

Equation 24 becomes singular at $D = \hat{\mu}_{max}$. Thus, as long as $\hat{\mu}_{max}$ is greater than μ_{max}, the optimization procedure is implementable. Therefore, it is necessary to have a lower bound on the value of $\hat{\mu}_{max}$. By using Eq. 24 to construct the modified equality constraint, $F(D,D^*,\hat{\psi},\psi^*)$, it is possible to reduce Eq. 21a to a quartic equation, which can be solved analytically. However, this is probably not typical for most processes, and we expect that Eq. 21a would usually have to be solved numerically.

Using this approximation for the steady-state behavior in the optimization scheme, we have incorporated model mismatch into the simulation in two ways.

1. We assumed the yield to be constant when it is actually a function of the dilution rate at steady state.

2. We assumed very different values for the kinetic parameters used in the optimization scheme ($\hat{\mu}_{max}$, $\hat{K}_s$, and $\hat{Y}$) than for those in the process being simulated.

It is also worth mentioning that the optimization algorithm does not require the knowledge of the initial substrate concentration, S_o, even though the optimum dilution rate does depend on this parameter. Figure 3 shows the steady-state input-output behavior given by Eqs. 23 and 24, and Figure 4 shows their corresponding steady-state performance indices. We observe from Figure 4 that the dilution rate which optimizes the performance index of the approximated steady-state behavior causes a washout steady state in the true process. Washout is a phenomenon peculiar to continuous culture systems. If the dilution rate exceeds the cell growth rate, a steady state will be reached such that there are no cells left in the fermentor ($x = 0$). In this

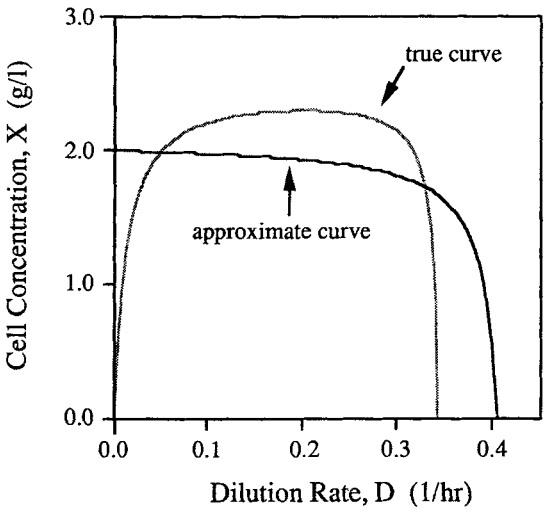

Figure 3. True and approximate steady-state cell concentration profiles.

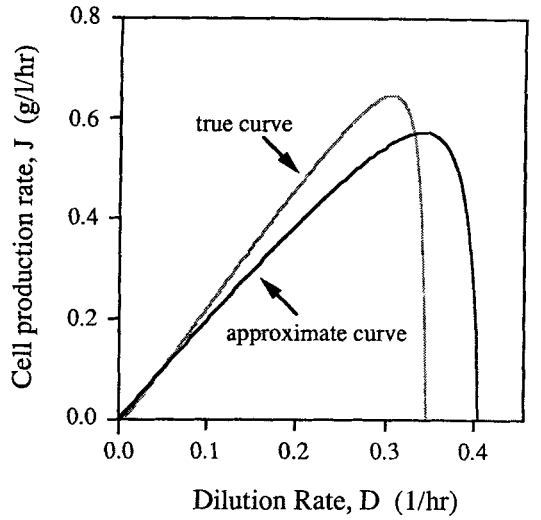

Figure 4. True and approximate performance index profiles.

respect, the approximated steady-state behavior is a poor representation of the true steady-state behavior of the process.

At this juncture, we will digress a moment to consider how the optimization algorithm would perform if there were perfect estimation of the curvature, tangent, and function value at the current operating point. In this case the steady-state manifold of the modified objective function can be found exactly, and thus, the curve, $\nabla_D \hat{J}(D,D^*) = 0$, can be identified as shown in Figure 5. Note that the upper curve, which crosses the operating line, $D_{opt} = D^*$, represents the input which is locally maximizing ($\nabla_D^2 \hat{J}(D,D^*) < 0$) at a given operating point, whereas the lower curve represents the input which is locally minimizing ($\nabla_D^2 \hat{J}(D,D^*) > 0$) at a given operating point. Over most of the operating range of the fermentor, the maximum input is the only physically realizable solution of the optimization algorithm.

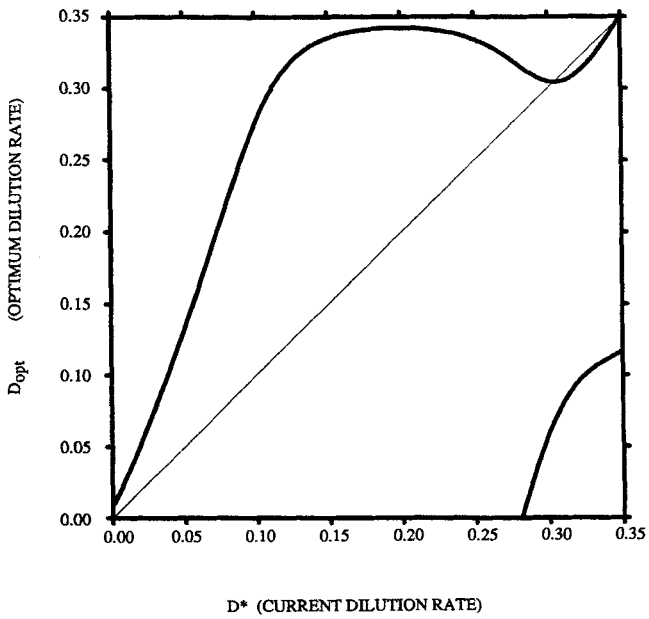

Figure 5. Optimality criterion for the modified objective function.

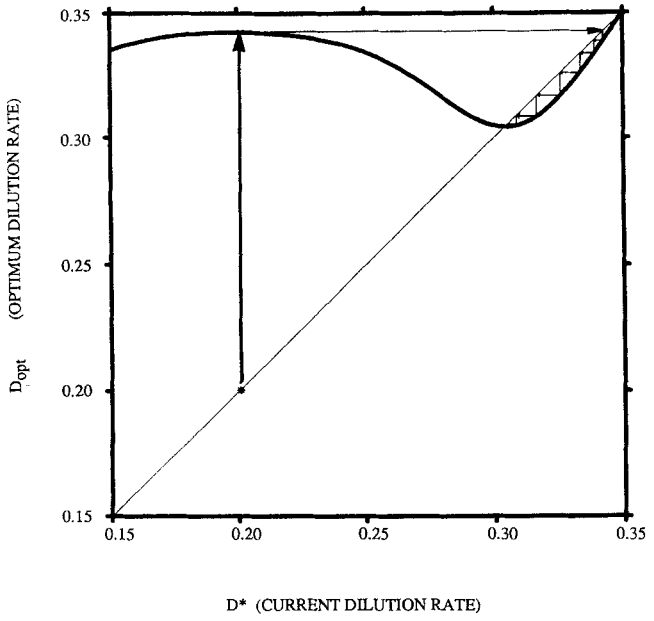

Figure 6. Path to optimum when there is perfect estimation.

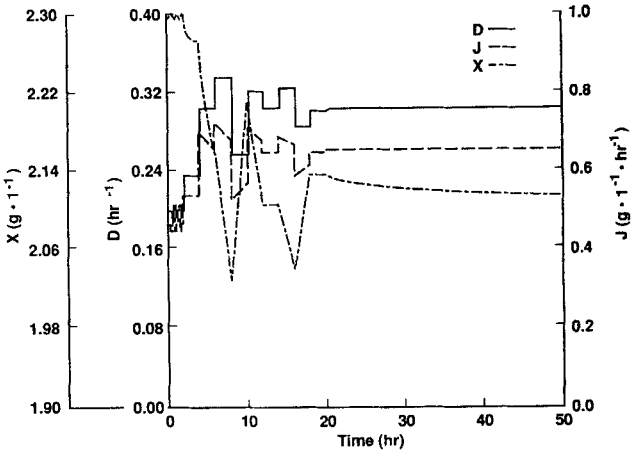

Figure 7. Simulation results: dilution rate, cell production rate, and cell concentration vs. time.

In the simulation to be discussed, the fermentor is initially operating at steady-state at a dilution rate of 0.2 h^{-1}, which corresponds to the "starred" point on Figure 6. The operating line ($D_{opt} = D^*$) and the curve, $\nabla_D \hat{J}(D,D) = 0$, intersect at the optimum dilution rate, $D = 0.3042\ h^{-1}$, which is the only stable fixed point of this map. The washout steady state, $D_{opt} = D^* = S_o\mu_{max}/(K_s + S_o) = .3438\ h^{-1}$, is also a fixed point, but it is unstable. By moving *vertically* from the operating line to the graph of $\nabla_D \hat{J}D,D^*) = 0$, we obtain the first "optimum" input predicted by the optimization algorithm. Then we move *horizontally* back to the operating line, symbolizing the implementation of this input. These two steps are repeated until the fixed point is reached as illustrated in Figure 6.

As we have seen in this example, if there is perfect estimation we have only one possible solution which maximizes the objective function considered. However, for other objective functions it is possible that more than one solution will optimize the modified objective function. In such a case, it would be necessary to define some heuristic rules for selecting which input to implement. If the equality constraints have singularities as in this example, one could implement a conservative input strategy by minimizing the difference between the current and "optimum" inputs. However, if there are no singularities, it is expeditious to implement the input which gives the overall optimum of the modified objective function.

These same input strategies are suggested when there is imperfect estimation since the curve, $\nabla_D \hat{J}(D,D^*) = 0$, which is now time-varying, may have more than one or no locally optimizing inputs in the operating range of the process. In the simulation and experimental results which follow, we solve a quartic equation to find candidates for the "optimum" dilution rate. Thus, there is the possibility of zero, one, or two "optimum" solutions. Therefore, we have developed two heuristic rules for the selection of the input in the case of nonuniqueness:

1) If there are two possible inputs, choose the input which minimizes the absolute value of the difference between the current input and the possible "optimum" inputs.

2) If there are no possible inputs, increment the current input in the predicted direction of the optimum.

The value of the increment in rule 2 is dependent upon the necessary tuneable parameters of the algorithm and will be discussed later. Note that rule 2 momentarily reduces the optimization procedure to a steepest descent algorithm.

The simulation results for the optimization are shown in Figure 7. The dilution rate is the input signal from the optimization algorithm. The fermentor was initially operating at a steady state with a dilution rate of 0.2 h^{-1} when the estimation and optimization algorithms were started at $t = 0$ hours. The dilution rate at the new and optimal steady state appears to be 0.305 h^{-1} which corresponds well to the calculated optimal dilution rate of 0.3042 h^{-1}. Comparing Figures 6 and 7, we note that the predicted and actual paths to the optimum dilution rate are different. This is due to the time-varying nature of the curve, $\nabla_D \hat{J}(D,D^*) = 0$, which results from the imperfect estimation of the necessary on-line process information, as well as the incorporation of an input constraint imposed at start-up, which will be discussed later.

Figure 7 also shows the cell production rate and the exiting cell concentration, respectively. We see that the optimal cell production rate of 0.64 g/L/h was reached in 20 hours or ten optimization intervals. The time required to reach the optimal cell production rate depends on how far the initial steady state is from the optimal steady state. We also observe that during the optimization, the fermentor is operating in a transient regime as is expected.

The values of the tuneable parameters used in the simulation are given in Table 1. A discussion of how to select these parame-

Table 1. Values of Tuneable Parameters

$n = m = 3$
$T = 15$ min
$T_{opt} = 2.0$ h
$T_{int} = 2.0$ h
$\Delta_{max} = [0.0175 \times (t - T_{int} + T_{opt})]\ h^{-1}$*

*Optimization began at $t = 2$ h.

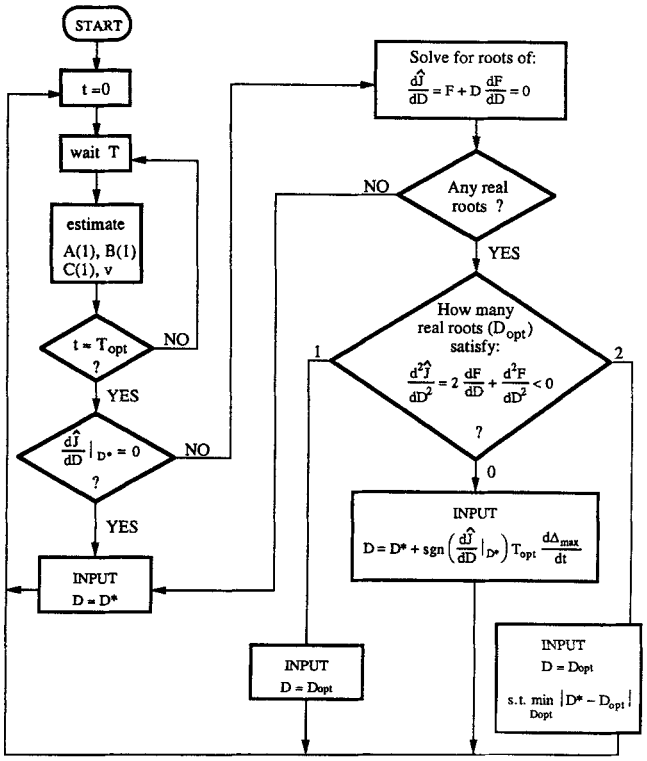

Figure 8. Optimization procedure for the continuous culture system.

ters follows the experimental results. A flow chart of the optimization procedure is shown in Figure 8.

Experimental Results

The optimization algorithm described in the previous sections was applied with a few modifications to a bench scale continuous fermentation process in which the yeast *candida utilis* was grown on a glucose substrate.

Candida utilis Y-1082 was grown in a 100 mL batch culture on yeast growth medium at 30°C, and 5 mL of the culture at the exponential phase were transferred to inoculate the continuous culture system. Two stock solutions were prepared and stored at 4°C: (A) 20 × $NH_4(PO_4)$, consisting of 100 g/L $(NH_4)_2SO_4$ and 40 g/L KH_2PO_4 and (B) 40 × Peptone, consisting of 20 g/L $MgSO_4$, 8 g/L NaCl, 8 g/L protease peptone, and 8 g/L yeast extract. The Yeast Growth Medium was prepared by mixing 175 mL of stock solution (A), 87.5 mL stock solution (B), and 17.5 g of dextrose. The feed glucose concentration was 5 g/L. A schematic of the continuous culture system with the operating conditions is shown in Figure 9.

The manipulated variable used in the experiment as in the simulation, is the dilution rate. Since the volume of the fermentor was kept constant, this translated into manipulating the feed flow rate of the medium, using a twenty-step variable speed pump. Similarly, as in the simulation, the cell concentration was the output variable and was measured on-line using a flow-through cell, based on the design of Lee and Lim (1980), in a spectrophotometer. The optimization algorithm was run on a digital computer, however, the pH and the water bath temperature were controlled using low-level proportional controllers.

In the optimization algorithm, we used, as in the simulation study, the Monod model, given by Eq. 23, as the approximate

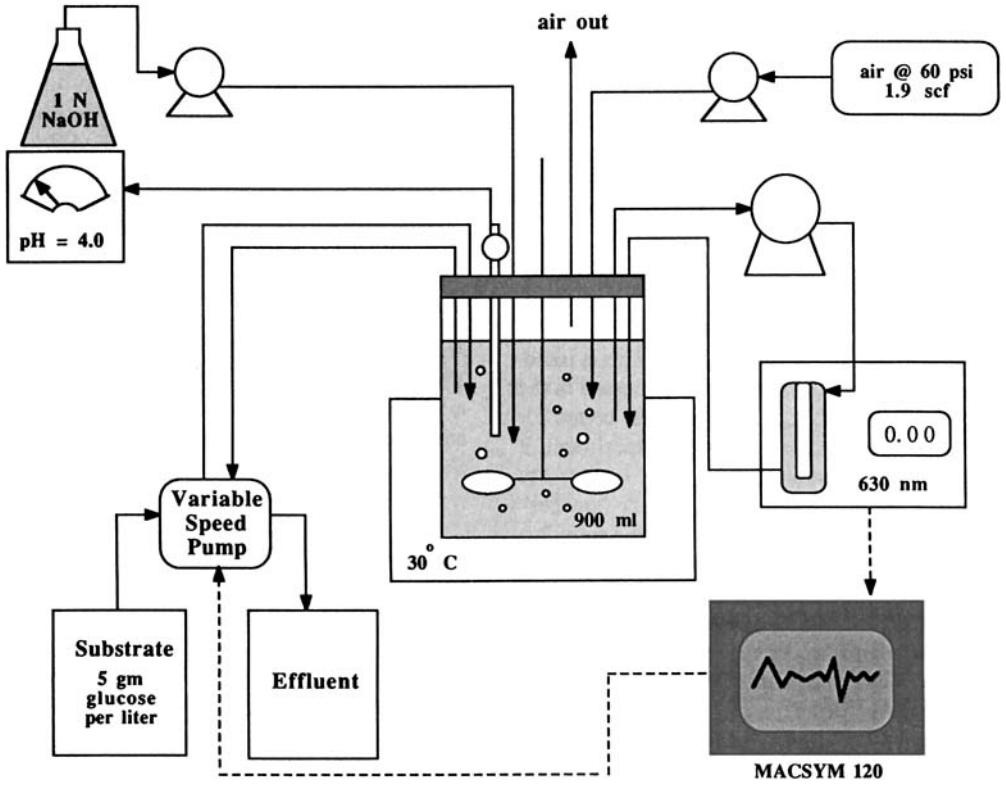

Figure 9. Continuous culture system with operating conditions.

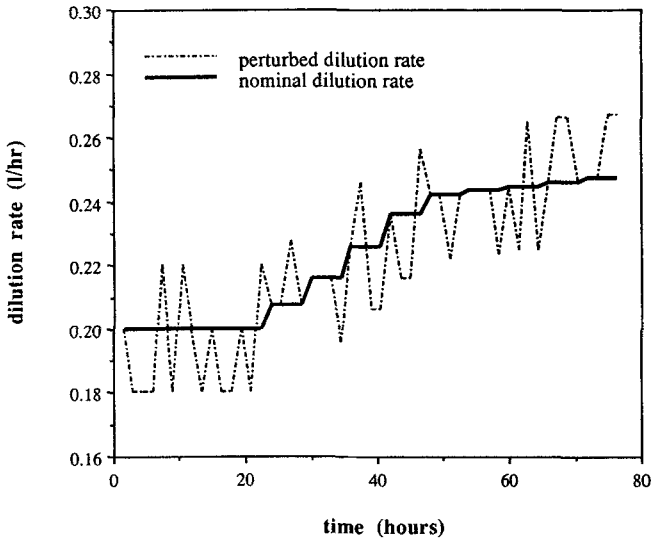

Figure 10. Experimental results: dilution rate vs. time.

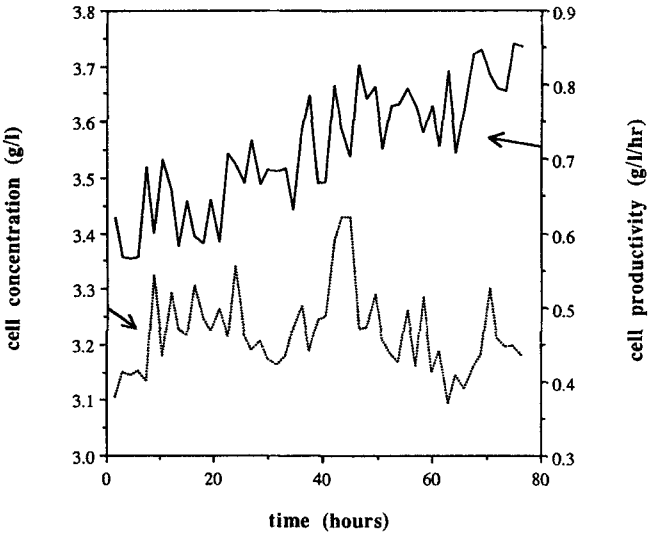

Figure 11. Experimental results: cell concentration and cell production rate vs. time.

model with

$$\hat{Y} = 0.4$$
$$\hat{K}_s = 0.09 \text{ g/L}$$
$$\hat{\mu}_{max} = 0.25 \text{ h}^{-1}$$

The values of $\hat{K}_s$ and $\hat{\mu}_{max}$ were determined from off-line batch experiments. The results from one representative experiment are shown in Figures 10 and 11. The tuneable parameters of the algorithm used in the experiment are given in Table 2.

The only differences between the implementation of the optimization algorithm in the simulation and in the experiment show up in the input signal. First, note that a much longer initialization period was used. This was only a precaution to ensure good initial estimates to the optimizer which was taken due to the length of each experimental run. Furthermore, a PTRS was superimposed on the input signal for the duration of the experiment. This was also a precaution to ensure that sufficiently exciting signals were sent to the estimator as discussed previously.

It is interesting to notice that the cell concentration does not change significantly during the execution of the experiment. However, the productivity of the fermentor is increased by approximately 20% due to the higher dilution rate. This is indicative of a broad optimum. Due to the singularity discussed in the simulation section, the parameter, $\hat{\mu}_{max}$, sets an upper bound for the allowable dilution rate for the system. In the experiment, it is clear that a somewhat higher value for $\hat{\mu}_{max}$ may have led to somewhat better results since the optimizer in fact "pushes" against this constraint. Another technique to avoid the singularity would be to recursively estimate the kinetic parameters online (Golden, 1988).

The most important fact to notice is that the inclusion of the off-line model allows the algorithm to converge to the optimal performance in a relatively short amount of time. In this particular case, the algorithm converged after about 30 hours excluding the initialization period which was extended to provide the optimizer with good initial estimates. All of the tuneable parameters including the length of the intialization period were chosen to give relatively sluggish performance and they can be tuned to get better performance.

Discussion of Tuneable Parameters

The on-line optimization procedures discussed in the introductory section of this paper use adaptive estimation but do not include any other information about the static nonlinearities of the process. They have all been shown to give acceptable results in both simulations and experiments. To implement these procedures, numerous tuning parameters must be either known a priori or determined in an initialization procedure, which can take 30 or more hours for a continuous culture system. The performance of the optimization algorithm depends critically on the choice of these parameters, and may diverge if they are not selected properly.

The optimization procedure developed in this paper includes the approximate static nonlinearities of the process. The inclusion of the static nonlinearity facilitates the design of an algorithm which requires few tuneable parameters. The time-varying parameters, $A(1)$, $B(1)$, $C(1)$, and v in Eq. 10 are found using a recursive least squares estimator with UDU^T factorization (Ljung and Söderström, 1983) which is given in detail in Appendix 2 and the variable forgetting factor given by Eq. 17. The orders of the A, B, and C polynomials, which are denoted by n and m respectively, are typically chosen to be greater than or equal to 3. The estimation algorithm is relatively insensitive to these tuning factors if they are chosen in this manner.

Table 2. Values of Tuneable Parameters

$n = m = 3$
$T = 90$ min
$T_{opt} = 6.0$ h
$T_{int} = 19.5$ h
$\Delta_{max} = [0.0017 \times (t - T_{int})]$ h^{-1}*

*Optimization began at $t = 24$ h.

The optimization procedure seems only moderately sensitive to the choice of the sampling period, T. One rule of thumb for adaptive control is that the sampling period should be approximately 1/5 the fastest time constant of interest (Goodwin and Sin, 1984). This rule of thumb does not appear to apply in our application. Since we are interested in static relationships, it seems to be more effective to choose sampling periods which are short relative to the process settling times to get faster parameter convergence. In fact we found that in the simulation and experiment it was best to use a sampling period of approximately 1/50–1/100 of the dominant time constant of the culture system.

The input signal from the optimization procedure changes every optimization interval, N, which is an integer multiple of the sampling period. This allows an identification phase to precede each change in the input signal. Thus, the parameters necessary for the optimization control strategy will depend more on the current operating point than on past process operating conditions (Sternby, 1980). In this way, we are assured good estimates of the steady-state information necessary for the optimization procedure. The length of the optimization interval, T_{opt}, is chosen to be somewhat shorter than the process settling time which is on the order of 10–20 hours for continuous culture systems. A good choice for the optimization interval in this application appears to be 1/10–1/5 of the process settling time. However, it appears that the optimization interval should be at least two times the largest model order ($2 \times \max\{n,m\}$) sampling periods long to adequately reduce the effect of past process information on the current parameters. This is a heuristic based on our simulation and experimental experience.

The least squares estimator requires excitation of the process input and output to ensure that accurate parameter estimates are obtained. In the case of the simulation of the continuous fermentor, we found that the excitation needed for parameter identification was provided by the typically large changes in the input signal generated by the optimization procedure. However, to ensure that the estimator has rich data initially, a three-level pseudorandom signal was superimposed on the control signal during an initialization interval of length, T_{int}, which is determined by the user but must be at least one optimization interval long. This superimposed signal, which varied between ±2.5% of the initial input, is visible in Figure 7. For the case of the experiment, a similar signal of ±10% of the initial input was continually superimposed upon the input signal generated by the optimization algorithm to ensure good estimates in the presence of measurement noise and disturbances in operating conditions (see Figure 10).

To ensure process stability during the start-up period of the on-line optimization procedure, we impose a constraint limiting the magnitude of the first few changes in the process input. This is done by making the maximum allowable change in the input a linear function of time. This is a technique often used in adaptive control to minimize the effect of the first few sets of parameter estimates on the control strategy in case they are very inaccurate. Thus, as our confidence in the estimates of the steady-state information going to the optimization algorithm increases so does the maximum allowable step size, Δ_{max}. The number of input signals affected by this constraint is determined by the slope associated with the maximum allowable change in the process input. This slope is therefore an additional tuning factor. Good choices for the slope are 10–25% of the process input at start-up divided by the length of the optimization interval. We can see from Figure 7 that the maximum allowable step change was effective in constraining the input signal for the first two step changes only in the simulation. In the experiment, this was true for only the first step change.

The product of the slope, $d\Delta_{max}/dt$, and the optimization interval, T_{opt}, is the increment used in heuristic rule 2 of the defined input strategy as shown in Figure 8. The choice of the increment was motivated by a desire to reduce the number of tuneable parameters. Note that this rule was never needed in the simulation results shown.

Finally, we point out that the initialization period is not needed to determine any of the tuneable parameters required to implement our optimization procedure. They were either determined from knowledge of the system or approximated from the kinetic parameters used in the steady-state design equation given by Eq. 23, which gives some indication of the process settling times and time constants. Furthermore, experimental results seem to indicate that the optimization algorithm is robust with respect to the selection of the tuneable parameters.

Summary and Conclusions

The paper describes an approach to adaptive extremum control which combines the use of *a priori* models with on-line process estimation. The *a priori* model contains the basic nonlinearities of the process and it may be obtained from a steady-state design relationship. A least squares estimator with a new update formula for the variable forgetting factor is used to identify a second order Hammerstein model from which estimates of local geometric characteristics of the process can be obtained. Using a simple relationship from differential geometry, we can construct a new static input-output relationship for the process. This relationship, which adequately describes the steady-state behavior of the fermentor about the current operating point, is used to obtain a modified objective function. The input which optimizes this objective function and satisfies the defined input strategy is implemented. This procedure is periodically repeated and the optimal steady state is eventually reached.

It is demonstrated in the paper that if a matching condition is satisfied and the data is persistently exciting then the control converges to optimal control provided that the algorithm is initialized in a region of convexity that includes the optimal control. A simple simulation example and an experiment are included to illustrate the performance of the adaptive extremum controller.

Acknowledgments

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Notation

$A(\cdot), B(\cdot), C(\cdot)$ = polynomials in q^{-1} in the second-order Hammerstein model
a_i, b_i, c_i = coefficients of polynomials $A(\cdot)$, $B(\cdot)$, and $C(\cdot)$
D = dilution rate, h^{-1}
D_{opt} = optimum dilution rate, h^{-1}

$e(t)$ = prediction error
$E(\cdot)$ = expected value
f = equality constraint
F = modified equality constraint
J = value of objective function evaluated at given operating condition
k_1, k_2 = constants in Eq. 6a
K_s = half saturation constant, g/L
L = objective function
m_c = cell maintenance coefficient, h^{-1}
m = model order of polynomials $B(\cdot)$ and $C(\cdot)$
n = model order of polynomial $A(\cdot)$
N = characteristic length of sequence which is persistently exciting; also corresponds to the number of sampling periods in optimization interval
$N(t)$ = recursive least squares objective function given in Eq. 14
p = equality constraint
$P(t)$ = covariance matrix
q^{-1} = backshift operator defined by $q^{-1}x(t) = x(t-1)$
$r(t)$ = estimate of measurement variance
R = model mismatch residual
S = substrate concentration, g/L
S_0 = initial substrate concentration, g/L
T = sampling period
T_{int} = length of initialization interval
T_{opt} = length of optimization interval
u = process input
u_s = PTRS sequence
u_{so} = magnitude of PTRS sequence
v = process bias
$V(t)$ = Lyapunov function candidate
$w(i)$ = sequence of positive weights given in Eq. 14
X = cell concentration, g/L
y = process output
Y = yield coefficient

Greek letters

γ_{min} = minimum eigenvalue
δ = a positive number in Assumption 1
Δ = model error
$\Delta_{max}(t)$ = maximum allowable change in process input
ϵ_1, ϵ_2 = constants which characterize the degree of persistent excitation
$\lambda(t)$ = variable forgetting factor
$\varphi(t)$ = regression vector
$\theta(t)$ = parameter vector
$\tilde{\theta}(t)$ = parameter error vector
ρ = dimension of vectors φ and θ
μ = specific growth rate, h^{-1}
μ_{max} = maximum specific growth rate, h^{-1}
$\Sigma(t)$ = asymptotic memory length
Ω = remainder term in Taylor series given in Eq. 8
ψ = vector of process parameters and variables

Superscripts

$*$ = current operating condition or point
o = optimal solution
T = transpose of vector
$\hat{}$ = approximation or estimate

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Appendix 1: Proof of Results

Proof of Theorem 1. By multiplying through with $P(t)^{-1}$ in step 2 of the algorithm we have

$$P(t)^{-1}\tilde{\theta}(t) = P(t)^{-1}\tilde{\theta}(t-1) - \varphi(t-1)e(t)/(\lambda(t)r(t) + \varphi(t-1)^T P(t-1)\varphi(t-1))$$

where $\tilde{\theta}(t) = \theta - \hat{\theta}(t)$ is the parameter error. Defining $V(t) = \tilde{\theta}(t)'P(t)^{-1}\tilde{\theta}(t)$, we obtain

$$V(t) = \lambda(t)V(t-1) - e(t)^2 \cdot [1 - \varphi(t-1)^T P(t)\varphi(t-1)/r(t)]/r(t) + R(t)^2/r(t)$$

By recursion on this equation

$$V(t) = w(t-N-1)V(t-N) - \sum_{i=t-N}^{t} w(i) \cdot \{e(i)^2(1 - \varphi(i-1)^T P(i)\varphi(i-1)/r(i))/r(i) - R(i)/r(i)\}$$

where $w(i)$ are the weights defined by multiplying forgetting factors. Using the boundedness property of the forgetting factor we now have

$$V(t) \le \lambda_{max}^N V(t-N) + R\frac{1-\lambda_{max}^N}{1-\lambda_{max}}$$

and then, using the fact that the regressor is persistently excited

we have

$$\|\tilde{\theta}(t)\|^2 \leq \frac{R}{(\lambda_{\min}^N \epsilon_1)} \cdot \frac{1-\lambda_{\max}^N}{1-\lambda_{\max}} + \|\tilde{\theta}(t-N\|^2 \lambda_{\max}^N |P(t-N)^{-1}|$$

where $|P(t-N)^{-1}|$ is the operator norm of the inverse of the covariance matrix at the start of the data segment. The result then follows by appropriate definitions of constants. QED.

The Lyapunov approach used to establish Theorem 1 was inspired by the approach developed by Solo (1979) for the approximate maximum likelihood algorithm. Solo investigated the problem of estimating the parameters of an autoregressive moving average model driven by white noise. This approach has merit when $R(t)$ has a significant stochastic colored noise component and it can be included in the algorithm discussed here in a likewise manner. The result given above follows as before by using the Martingale convergence theorem. However, in our application and simulation studies we have not found much merit in estimating the moving average noise dynamics. The sampling rate is very slow and low pass filters effectively reduce the effect of random noise in the measurements. The remaining errors were found to be uncorrelated in time. The standard least squares estimator then gives unbiased estimates.

Proof of Result 1. Define $J = L(u, f(u))$ and $\hat{J} = L(u, F(u, u^*))$. Then

$$F(u^*, u^*) = f(u^*)$$

$$\left.\frac{dF}{du}\right|_{u^*} = \left.\frac{df}{du}\right|_{u^*}$$

$$\left.\frac{d^2F}{du^2}\right|_{u^*} = \left.\frac{d^2f}{du^2}\right|_{u^*}$$

By definition:

$$\nabla_u J = \left.\frac{\partial L}{\partial u}\right|_f + \left.\frac{\partial L}{\partial f}\right|_u \left(\frac{df}{du}\right)$$

$$\nabla_u \hat{J} = \left.\frac{\partial L}{\partial u}\right|_F + \left.\frac{\partial L}{\partial F}\right|_u \left(\frac{dF}{du}\right)$$

Thus

$$\nabla_u J - \nabla_u \hat{J} = \left.\frac{\partial L}{\partial u}\right|_f - \left.\frac{\partial L}{\partial u}\right|_F + \left.\frac{\partial L}{\partial f}\right|_u \left(\frac{df}{du}\right) - \left.\frac{\partial L}{\partial F}\right|_u \left(\frac{dF}{du}\right)$$

Furthermore

$$\left.\frac{\partial L}{\partial u}\right|_{f(u^*)} = \left.\frac{\partial L}{\partial u}\right|_{F(u^*)}$$

and

$$\left.\frac{\partial L}{\partial f}\right|_{u^*} = \left.\frac{\partial L}{\partial F}\right|_{u^*}$$

Thus

$$\nabla_u J(u^*) = \nabla_u \hat{J}\ (u^*, u^*)$$

But

$$\nabla_u \hat{J}\ (u^o, u^*) = 0$$

If the series $\{u_i^o\}$ converges to an arbitrary u^o, then

$$\nabla_u \hat{J} = 0 \text{ at } u = u^o = u^*$$

and

$$\hat{J} = L(u^*, F(u^*, u^*)) = L(u^o, F(u^o, u^o))$$

Thus

$$\nabla_u J = 0 \text{ at } u = u^o = u^*.$$

QED.

Appendix 2: Least Squares Estimation Algorithm Using the UDU^T Factorization

Initialize:

$D = 1{,}000 * I$

$U =$ an upper unit triangular matrix, typically $U = I$

$$f = U^T(t-1)\varphi(t)$$

$$g = D(t-1)f$$

$$\beta_0 = \lambda(t)$$

For

$$j = 1 \text{ to } \rho$$

$$\beta_j = \beta_{j-1} + f_j g_j$$

$$D(t)_{jj} = \beta_{j-1} D(t-1)_{jj} / \beta_j \lambda(t)$$

$$\nu_j = g_j$$

$$\mu_j = -f_j / \beta_{j-1}$$

For

$$i = 1 \text{ to } j-1 \quad (\text{if } j = 1, \text{ next } j)$$

$$U(t)_{ij} = U(t-1)_{ij} + \nu_i \mu_j$$

$$\nu_i = \nu_i + U(t-1)_{ij} \nu_j$$

Next i
Next j

For

$$i = 1 \text{ to } \rho$$

$$L_i = \nu_i / \beta_\rho$$

$$\theta(t)_i = \theta(t-1)_i + L_i e(t)$$

Next i

where

ρ = number of parameters being estimated $(n + 2m + 3)$
$\lambda(t)$ = variable forgetting factor updated using Eq. (17)
$e(t)$ = prediction error

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